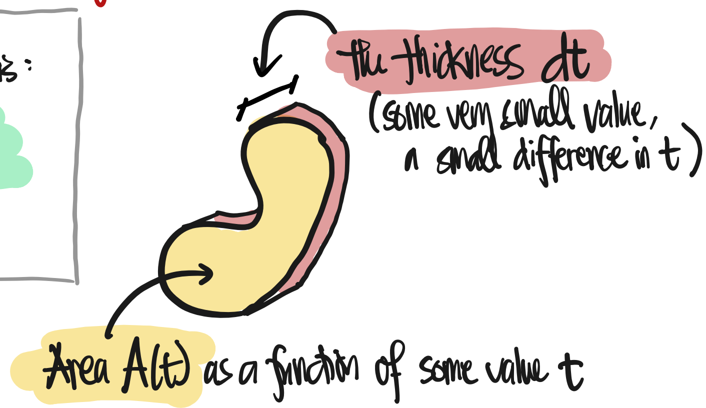
Calculating Volumes of Solids by Slicing.

Key Idea: Partition a solid into slices that are extrusions / cylinders of some cross-sectional area.

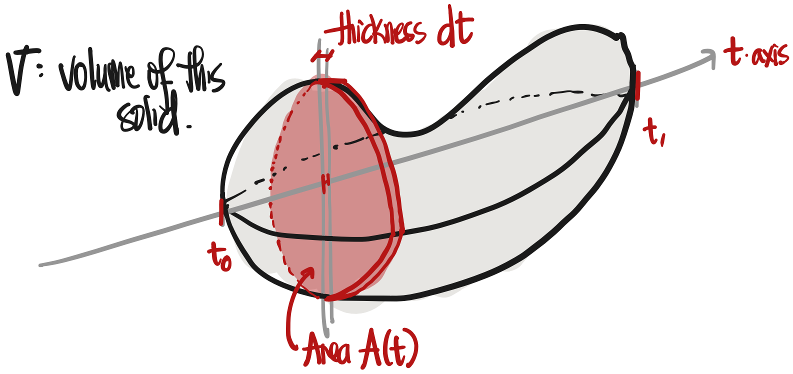
The volume dV of an extrusion is:

$$dV = (\text{area})(\text{thickness})$$
$$= A(t)\ dt$$



So, the volume V of a solid is

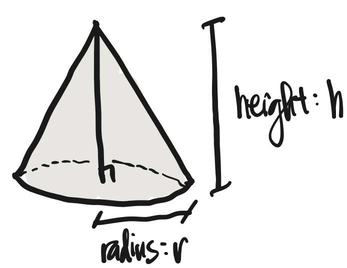
$$V = \left(\begin{matrix}\text{sum of the volumes of}\\ \text{all these slices}\end{matrix}\right) = \int dV = \int_{t_0}^{t_1} A(t)\ dt \quad \text{where } t \in [t_0, t_1]$$



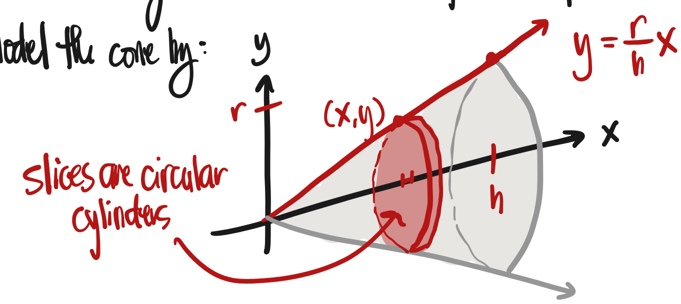
Remarks: ★ For modeling solids, the t-axis is usually the x-axis or y-axis. This will depend on the problem.

★ Usually, the cross-sectional area $A(t)$ with respect to the t-axis is given or enough information is provided to identify $A(t)$.

Example. Use the slicing method to derive the formula for the volume V_{cone} of a cone



Model the cone by:



Slices: cross sectional area (circles): $A = \pi y^2$ with $y = \frac{r}{h}x$

thickness: dx

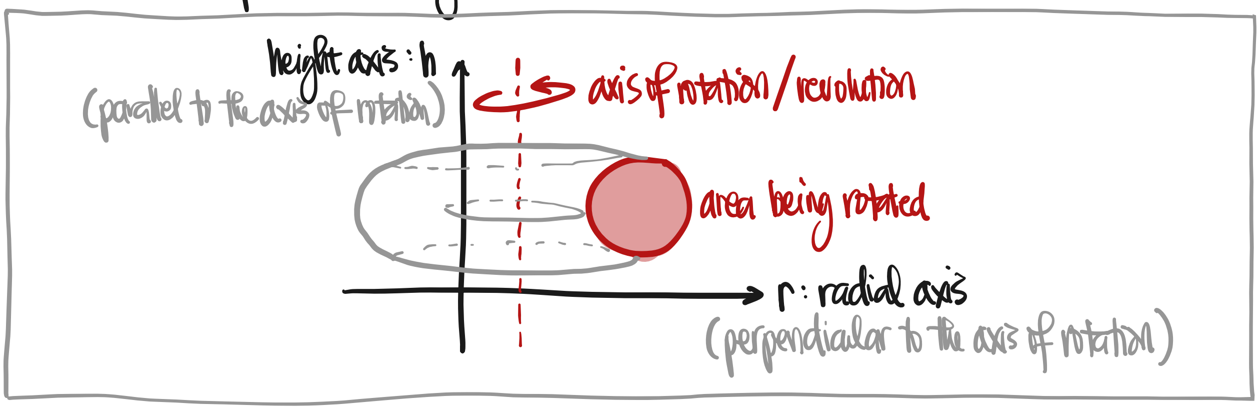
$$dV = \pi y^2\, dx = \pi\left(\frac{r}{h}x\right)^2 dx \quad \text{over } x \in [0,h]$$

; Volume of the cone:

$$V = \int_{x=0}^{x=h} \frac{\pi r^2}{h^2} x^2\, dx = \frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$
$$= \frac{\pi r^2}{h^2}\left(\frac{1}{3}h^3 - 0\right) = \frac{1}{3}\pi r^2 h \,;$$

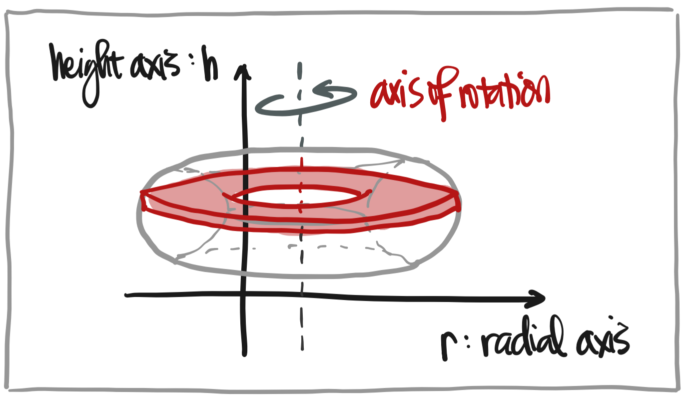
If the solid is a **solid of revolution**, we can calculate the volume either using
① the disk/washer method or ② the cylindrical shells method.
Each method slices the solid of revolution in a specific manner.

We model the solid of revolution using two axes:



① The Disk/Washer Method

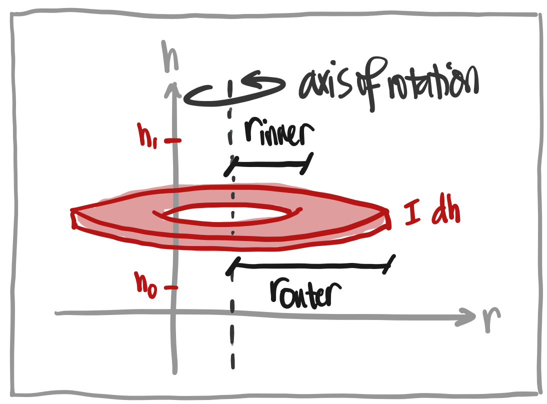
slices the solid **perpendicular** to the axis of rotation.



There are 4 things to find:

① the thickness dh (this is either dx or dy)
② the interval $[h_0, h_1]$ on the height axis that covers the solid.
③ the outer radius $r = r_{outer}$ as a function of h
④ the inner radius $r = r_{inner}$ as a function of h

these depend on the axis of rotation!
these also have to be positive!



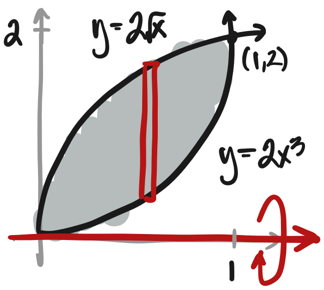
Then: volume of slice: $dV = \underbrace{\pi r_{outer}^2 \, dh}_{\text{volume of outer disk}} - \underbrace{\pi r_{inner}^2 \, dh}_{\text{volume of inner disk}} = \pi (r_{outer}^2 - r_{inner}^2) \, dh$

volume of solid: $\boxed{V = \int_{h_0}^{h_1} \pi (r_{outer}^2 - r_{inner}^2) \, dh}$

TIP: When doing these types of problem, sketch the rotated region + strip and explicitly identify the required axis of rotation!

Examples using the Disk/Washer Method.

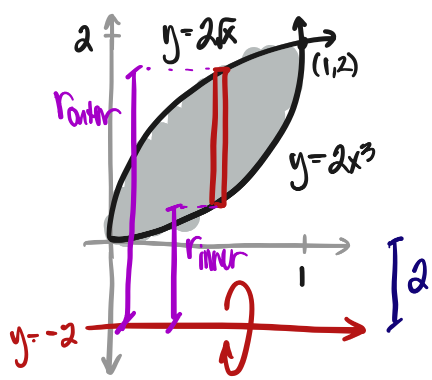
① Base : Region bounded by $y=2\sqrt{x}$ and $y=2x^3$;
Axis of Rotation : about the x-axis.



thickness: dx
bounds : $x \in [0,1]$
outer radius: $r_{outer} = y$ with $y = 2\sqrt{x}$; $r_{outer} = 2\sqrt{x}$
inner radius: $r_{inner} = y$ with $y = 2x^3$; $r_{inner} = 2x^3$

$$V = \int_0^1 \pi\left[(2\sqrt{x})^2 - (2x^3)^2\right] dx = \cdots = \frac{10}{7}\pi \, ;$$

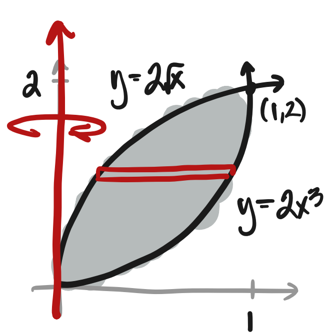
② Base : Region bounded by $y=2\sqrt{x}$ and $y=2x^3$
Axis of Rotation : about $y=-2$



thickness: dx
bounds : $x \in [0,1]$
outer radius: $r_{outer} = y+2$ with $y=2\sqrt{x}$; $r_{outer} = 2\sqrt{x}+2$;
inner radius: $r_{inner} = y+2$ with $y = 2x^3$; $r_{inner} = 2x^3+2$;

$$V = \int_0^1 \pi\left[(2\sqrt{x}+2)^2 - (2x^3+2)^2\right] dx = \cdots = \frac{100}{21}\pi \, ;$$

③ Base : Region bounded by $y=2\sqrt{x}$ and $y=2x^3$
Axis of Rotation : about the y-axis.

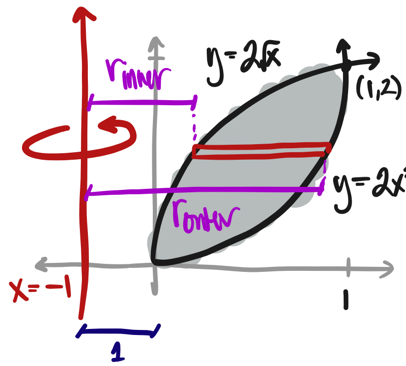


thickness: dy
bounds : $y \in [0,2]$
outer rad: $r_{outer} = x$ with $y = 2x^3$; $\frac{1}{2}y = x^3$; $x = \sqrt[3]{\frac{1}{2}y} = r_{outer}$;
inner rad: $r_{inner} = x$ with $y = 2\sqrt{x}$; $y^2 = 4x$; $x = \frac{1}{4}y^2 = r_{inner}$;

$$V = \int_0^2 \pi\left[\left(\sqrt[3]{\tfrac{1}{2}y}\right)^2 - \left(\tfrac{1}{4}y^2\right)^2\right] dy = \cdots = \frac{4}{5}\pi \, ;$$

④ Base : region bounded by $y = 2\sqrt{x}$ and $y = 2x^3$

Axis of Rotation : about $x = -1$



Thickness : dy

bounds : $y \in [0,2]$

outer rad. : $r_{outer} = x + 1$ with $y = 2x^3$; $x = \sqrt[3]{\frac{1}{2}y}$;

$r_{outer} = \sqrt[3]{\frac{1}{2}y} + 1$;

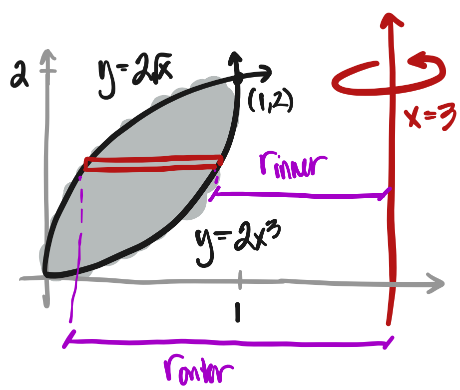
inner rad : $r_{inner} = x + 1$ with $y = 2\sqrt{x}$; $x = \frac{1}{4}y^2$;

$r_{inner} = \frac{1}{4}y^2 + 1$;

$$V = \int_0^2 \pi\left[\left(\sqrt[3]{\tfrac{1}{2}y} + 1\right)^2 - \left(\tfrac{1}{4}y^2 + 1\right)^2\right] dy = \cdots = \frac{37}{15}\pi \ ;$$

⑤ Base : region bounded by $y = 2\sqrt{x}$ and $y = 2x^3$

Axis of Rotation : about $x = 3$



thickness : dy

bounds : $y \in [0,2]$

outer rad : $r_{outer} = 3 - x$ with $y = 2\sqrt{x}$; $x = \frac{1}{4}y^2$;

$r_{outer} = 3 - \frac{1}{4}y^2$;

inner rad : $r_{inner} = 3 - x$ with $y = 2x^3$; $x = \left(\frac{1}{2}y\right)^{\frac{1}{3}}$;

$r_{inner} = 3 - \left(\frac{1}{2}y\right)^{\frac{1}{3}}$;

$$V = \int_0^2 \pi\left[\left(3 - \tfrac{1}{4}y^2\right)^2 - \left(3 - \left(\tfrac{1}{2}y\right)^{\frac{1}{3}}\right)\right] dy = \cdots = \frac{21}{5}\pi \ ;$$